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Critical exponent for the quantum Hall plateau transition

By

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Abstract

In this article we will briefly review our work on estimating the critical exponent of the Anderson transition at the centre of a Landau level. We review some basic facts about the quantum Hall effect and briefly describe how the critical exponent is measured. We explain why physicists think critical exponents are important. We also explain how the exponent can be estimated numerically and to what extent our current estimate is in agreement with experiments.

The quantum Hall effect [1] is the very precise (up to a few parts per billion) quantization of the Hall conductivity that is observed in a two-dimensional electron gas subjected to a large perpendicular magnetic field at very low temperatures

$$\sigma_{xy} = n\frac{e^2}{h} \tag{1}$$

Here, n is an integer. When measured as a function of magnetic field, the quantization is manifest as a series of plateaus in the Hall conductivity.

In the next few paragraphs we summarise some basic facts about the quantum Hall effect. We refer the reader to the review of Huckestein [2] for further details.

In a clean two dimensional electron gas in a strong perpendicular magnetic field the kinetic energy of the electrons would be exactly quantized

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$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c \tag{2}$$

Here, ω_c is the cyclotron frequency. These energy levels are referred to as Landau levels. The degeneracy of a Landau level is equal to the ratio of the total magnetic flux through the system to the flux quantum.

Disorder plays a crucial role in the quantum Hall effect in two ways. The first is to broaden the Landau level so that the energy quantization is no longer exact. The second is to cause Anderson localisation of the electron wave functions. This occurs for all states except those at the centre of a Landau level. Quantized values of the Hall conductivity are observed when the states at the Fermi level are localized. Transitions between the quantized values of the Hall conductivity occur when the Fermi level is driven through the delocalised states at the centre of a Landau level.

In experiments, the transitions between quantized values of the Hall conductivity are observed to become sharper at lower temperatures. The standard explanation for this phenomenon is as follows. Near the centre of the Landau level the localisation length has a power law divergence that is described by a critical exponent ν

$$\xi \sim |x - x_c|^{-\nu} \tag{3}$$

Here, x is the parameter, such as the applied magnetic field, that is used to drive the system through the transition point and x_c is the critical value of the parameter at which the transition occurs. As we approach the centre of a Landau level the localisation length grows. The transition between quantized values of the Hall conductivity occurs when the localisation length exceeds the phase coherence length ℓ_φ. This is the length scale on which the entanglement of the wavefunction of a single electron with the other electrons in the sample can be neglected. Since Anderson localisation is an interference phenomena, once the localisation length becomes longer than the phase coherence length, interference is suppressed and diffusion is restored. The phase coherence has a power law divergence at zero temperature.

$$\ell_\varphi \sim T^{-1/z} \tag{4}$$

This defines another critical exponent, the dynamic exponent z. Thus, at lower temperatures, the magnetic field interval over which the localisation length exceeds the phase coherence length is narrower and the transition sharper. Moreover, combining (3) and (4) we conclude that measurements of the Hall conductivity (or in practice the Hall resistance R_{xy}) as a function of magnetic field (over the interval around the transition

between quantized values) for different temperatures will yield a series of curves that cross approximately at a common critical value of the magnetic field with a slope at the critical field that varies with temperature as

$$\frac{dR_{xy}}{dB} \sim T^{-\kappa} \tag{5}$$

Here the exponent κ is related to the product of the critical exponent and the dynamic exponent

$$\kappa = \frac{1}{\nu z} \tag{6}$$

This is exactly what is observed in experiment [3].

Why do physicists place such importance on the value of critical exponents? The answer is universality (see [4] for a much more general discussion of this concept). The plateau transition in the quantum Hall effect has been observed in several different experimental systems; Silicon MOSFETs, semiconductor hetero structures and more recently graphene [5, 6]. These systems share similarities, for example, they are all effectively two-dimensional, but there are also many differences. Particularly in graphene, the Landau level structure is quite different from that in heterostructures or MOSFETs. Yet in each system, the same critical behaviour, and the same values of the critical exponent are observed. The renormalisation group theory of continuous phase transitions tells us that the values of critical exponents should be independent of the details of the experimental system and depend only on the dimensionality and the fundamental symmetries of the system (such as time reversal symmetry and spin rotation symmetry).

The phenomenon of universality is very convenient for the theorist. It means that the value of the critical exponent can be estimated using a very simplified model that neglects many experimental details, while still expecting agreement with the measured experimental value. In particular, agreement between experimental and theoretical values of the exponent means that the essential physics of the transition has been properly understood.

While, in principle, it should be possible to calculate critical exponents analytically, the standard perturbation techniques such as the epsilon expansion have not proved successful when applied to the Anderson transition. The gap has been filled by numerical estimations of the exponents [7, 8, 9, 10].

Critical exponents are estimated numerically by applying the finite size scaling method to the analysis of data from numerical simulations. In this approach, some physical quantity, call it Q, is estimated by simulation of the numerical model for various values of a control parameter x and for various system sizes L. The numerical

data are then fitted to a finite size scaling law of the form

$$Q = F\left(uL^{1/\nu}\right) \tag{7}$$

Here, u is a relevant scaling variable,

$$u \equiv u\left(x - x_c\right) \tag{8}$$

F is a scaling function and we have assumed that Q is dimensionless. In practice, in high precision work, it is necessary to take account of corrections to this finite size scaling law due to nonlinearity of the scaling variables and the presence of irrelevant scaling variables. We refer the reader to our paper [7] for details and references.

Our numerical analysis of the plateau transition has been performed using the Chalker-Coddington network model [11, 12]. This model is an idealisation of the situation in which a very large magnetic field is applied perpendicular to a two-dimensional electron gas. The magnetic field is supposed sufficiently large that mixing of different Landau levels can be completely neglected, and also that the magnetic length is much shorter than the correlation length of the random potential in the sample, so that the random potential appears smooth on this length scale. Under such conditions the motion of the electron consists of a rapid cyclotron motion with a drift of the cyclotron centre along the equipotentials of the random potential. The wavefunctions of the electrons are thus supported on the equipotentials. Tunnelling of electrons between equipotentials can occur at saddle points of the random potential, with a probability that reaches a maximum of one half at the centre of the Landau level. In the Chalker-Coddington model this system is replaced by a network of nodes, corresponding to saddle points, and links corresponding to equipotentials. The only random elements that remain are uniformly distributed random phases that model the phase picked up by an electron as it propagates along an equipotential between saddle points. The results of previous numerical simulations of this model are consistent with all the states in the Landau level being localised except exactly at the centre of the Landau level, where the localisation length diverges, i.e. they are consistent with the existence of an isolated critical fixed point surrounded by a localised phase.

In principle, the physical quantity Q calculated in the numerical simulations can be any physical quantity that is sensitive to the nature, localised or extended, of the wave functions. This includes quantities such as the inverse participation ratio, energy levels statistics, two terminal conductance etc. In practice, we have found that the most precise results are obtained by formulating the problem as a transfer matrix product and estimating the Lyapunov exponents of this product. The Lyapunov exponents are estimated by simulating a very long quasi-1D system. The role of the system size in (7) is played by the number of nodes in the cross-section and the parameter x is the energy

measured relative to the centre of the Landau level (so that the critical point is $x_c = 0$). The dimensionless quantity Q is usually taken to be the product of the smallest positive Lyapunov exponent γ and the number of nodes in the cross section of the network N.

$$Q = \gamma N \tag{9}$$

Data from such a simulation together with an appropriate finite size scaling fit (taking account of corrections to scaling) are shown in Figure 1. (For further details of the simulation we refer the reader to our paper [10].) Periodic boundary conditions are imposed on the network in the transverse direction so that the data are symmetric (within the statistical precision) around the centre of the Landau level. The data exhibit an increasing curvature as the number of nodes N increases, consistent with (7). For energies other than the centre of the Landau level, Q increases with N, which is characteristic of localised states. At the centre of the Landau level, scale invariance of Q is observed, which is characteristic of critical states. The estimate of the exponent obtained by careful fitting of the data is

$$\nu = 2.593\,[2.587, 2.598] \tag{10}$$

The numbers in bracket are a 95% confidence interval. Our estimate is consistent with the original work of Chalker and Coddington [11], but significantly larger than the estimate of Huckestein [2]. This may be explained by the fact that Huckestein simulated a different model, which would indicate non-universality of the exponent. However, we consider it more likely that Huckestein underestimated the exponent and the error bars.

So how well does our numerical estimate of the exponent agree with the measured value in experiments on the quantum Hall effect? The most recent experimental measurement is that of Li *et al.* [13, 14]. In this experiment measurements were performed as a function of magnetic field and temperature. Analysis of this data yielded the following estimate

$$\kappa = 0.42 \pm .01 \tag{11}$$

To extract a value for ν an independent estimate of z is needed. Li *et al.* obtained this by looking at the size dependence of measurements of the Hall resistance on very narrow Hall bar samples. They found

$$z \approx 1 \tag{12}$$

Unfortunately, they did not report a standard error for this result. Assuming that all the error is in ν and not z, we extract from their results the estimate $\nu = 2.38 \pm .06$. Our numerical estimate is significantly greater than this experimental value.

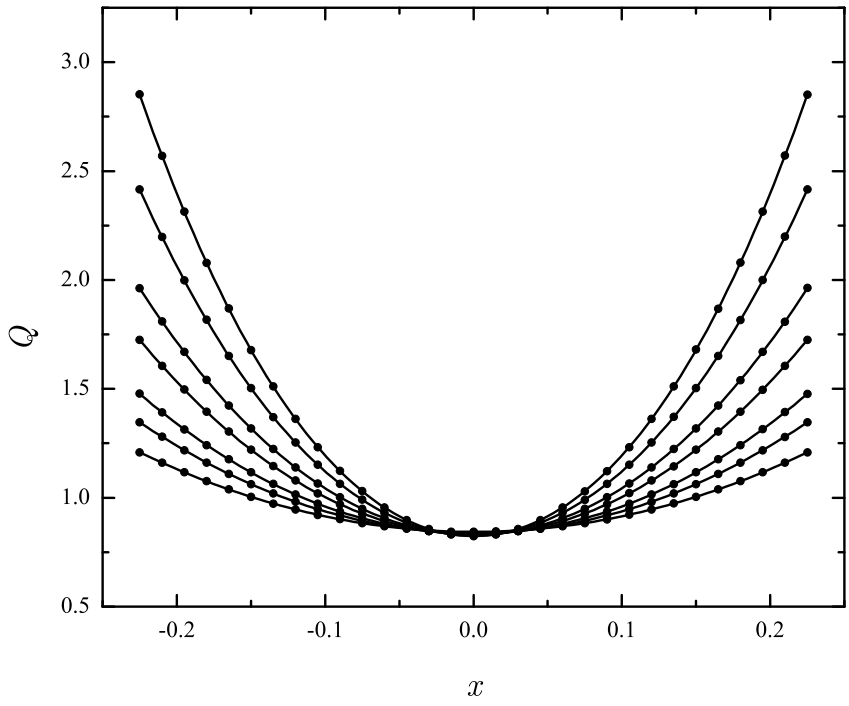

Figure 1. Finite-size scaling fit of numerical data obtained in a simulation of the Chalker-Coddington model. Different curves correspond to different numbers of nodes N in the cross section $N = 16, 24, 32, 48, 64, 96, 128$. Increasing curvature corresponds to increasing N. The numerical data have a precision of 0.03%.

So what might explain the discrepancy between the experimental and theoretical values of the critical exponent? In our opinion an important clue is provided by the value of the dynamic exponent. Models of non-interacting electrons predict $z = 2$ (or more generally $z = d$ where d is the dimensionality [15]) in clear disagreement with experiment. Simulations within the Hartree-Fock approximation suggest that the observed value of the dynamic exponent may be better explained by including the Coulomb interaction between the electrons [16]. If that is correct, there is no reason to suppose that the value of the critical exponent would be unchanged by the inclusion of the Coulomb interaction. Note that the point is not whether real electrons are interacting but whether such interactions are relevant at the non-interacting fixed point. The fact

that the results of Hartree-Fock simulations are consistent with a value of the dynamic exponent different from two suggests that interactions are relevant. More precise experimental measurement of the dynamic exponent, together with more precise simulations within the Hartree-Fock approximation, may shed light on the issue. However, both of these are likely to prove very challenging in practice.

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